Appendix A

General Overview of the n-Body Problem

A.1 Statement of the n-Body Problem and First Integrals

In this Appendix *some fundamental concepts and results* concerning the so-called *n-body problem* are presented. Most content of this Section is based on a part of Chapter III of the lecture notes *"Curso de Mecánica Celeste"*, [7].

The *n-body problem* is the problem which considers n bodies ($n \geq 2$) moving in $\mathbb{R}^3$ under the influence of their mutual interactions. For the purposes of this Undergraduate Dissertation, the following *general hypotheses* will be assumed:

- The bodies of the system under consideration will be idealized as *point particles* $P_1, \ldots, P_n$, with respective masses $m_1, \ldots, m_n$.
- A *Cartesian reference frame* is fixed in an Euclidean space $\mathbb{R}^3$, with its origin O at the center of mass (or barycenter) of the system, and a certain time parameter t (*absolute time* of the Newtonian Mechanics) is chosen. A spatial reference frame with its origin at the center of mass of the system is called a *barycentric reference frame*, or *barycentric coordinate system*. With respect to such a space and time reference frame, the instantaneous positions of the points $P_1, \ldots, P_n$ along time are located by means of the respective vectors $\overrightarrow{OP}_1 = \boldsymbol{r}_1(t), \ldots, \overrightarrow{OP}_n = \boldsymbol{r}_n(t)$, whose respective norms $\|\boldsymbol{r}_1(t)\|, \ldots, \|\boldsymbol{r}_n(t)\|$ represent their instantaneous distances from the origin of the spatial reference frame.
- The mutual interactions between two points P_i and P_k will be supposed to be a *central force*. The direction of that force is given by the line segment bounded by these two points, and its magnitude is given by $m_i\, m_k\, |f(\|\boldsymbol{r}_{ik}\|)|$, with $\|\boldsymbol{r}_{ik}\|$ the mutual distance between P_i and P_k.
- If we observe the vector $\boldsymbol{r}_{ik} = \boldsymbol{r}_k - \boldsymbol{r}_i$, it is easy to realize that the function $f(\|\boldsymbol{r}_{ik}\|)$ is related to the force that the body P_k exerts on the body P_i. The sign of this function is positive if we are dealing with a *repulsive force*, and negative if we consider an *attractive force*.

Under these general hypotheses, *the vector differential equation of motion of the body P_i* is given by

$$m_i\, \ddot{\boldsymbol{r}}_i = \sum_{k=1}^{n} m_i\, m_k\, \phi(\|\boldsymbol{r}_{ik}\|)\, \boldsymbol{r}_{ik}, \quad i = 1, \ldots, n, \tag{A.1.1}$$

where $\phi(r) = f(r)/r$, and the *dot notation* is used to represent derivatives with respect to t.

From now on, we will consider only the case $f(r) = r^{\lambda}$, with $\lambda \in \mathbb{R}$. The case $f(r) = \mathcal{G}\, r^{-2}$, where $\mathcal{G}$ is the universal gravitational constant, corresponds to the Newton's gravitational force. Under these conditions, the equations (A.1.1) become

$$m_i\, \ddot{\boldsymbol{r}}_i = \sum_{k=1}^{n} \mathcal{G}\, m_i\, m_k\, \frac{\boldsymbol{r}_{ik}}{\|\boldsymbol{r}_{ik}\|^2}, \quad i = 1, \ldots, n, \tag{A.1.2}$$

and govern the so-called *gravitational n-body problem.*

Both (A.1.1) and (A.1.2) are systems of n vector second-order ordinary differential equations, and give rise to respective differential systems of order $6\,n$. To solve them, we can think about different ways of addressing the question. We can search for $6\,n$ functionally independent first integrals, which allow us to solve the problem replacing variables; but for $n \geq 3$ it is not possible, since we cannot find the number of first integrals that we need. For this reason, the study of *certain particular solutions* allowing one to obtain some additional knowledge about the behaviour of the system is a useful approach.

Definition A.1. The *kinetic energy* $T = T(\dot{\boldsymbol{r}}_1, \ldots, \dot{\boldsymbol{r}}_n)$, the *force function* $U = U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n)$ and the *moment of inertia* $J = J(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n)$ of the system are the scalar functions

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i \, (\dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i) \,, \quad U = - \sum_{1 \leq i < k \leq n} m_i \, m_k \int f(\|\boldsymbol{r}_{ik}\|) \, \mathrm{d}\|\boldsymbol{r}_{ik}\| \,, \quad J = \sum_{i=1}^{n} m_i \, (\boldsymbol{r}_i \cdot \boldsymbol{r}_i) \,. \tag{A.1.3}$$

The *potential* V from which the force is derived is defined as $V = -\,U$.

Remark. For the case $f(r) = r^{\lambda}$, the preceding definition of U reads

$$U = - \sum_{1 \leq i < k \leq n} m_i \, m_k \, \frac{\|\boldsymbol{r}_{ik}\|^{\lambda+1}}{\lambda + 1}. \tag{A.1.4}$$

Proposition A.1. *The equations (A.1.1), with U given by* (A.1.4)*, can also be rewritten as*

$$m_i \, \ddot{\boldsymbol{r}}_i = \nabla_{\boldsymbol{r}_i} U, \quad \text{where} \quad \nabla_{\boldsymbol{r}_i} U = \sum_{k} m_i \, m_k \, \|\boldsymbol{r}_{ik}\|^{\lambda-1} \, \boldsymbol{r}_{ik}, \quad i = 1, \ldots, n. \tag{A.1.5}$$

Proposition A.2. *The following equality holds:*

$$\boldsymbol{r}_i \cdot \nabla_{\boldsymbol{r}_i} U = (\lambda + 1) \, U. \tag{A.1.6}$$

Proof. Since U is a homogeneous function of degree $\lambda + 1$, by virtue of Euler's theorem, Equation (B.3.2), Appendix B, the equality holds. ∎

Proposition A.3. *J satisfies the* Lagrange-Jacobi identity*, namely*

$$\ddot{J} = 4T + 2\,(\lambda + 1)U. \tag{A.1.7}$$

Proof. Differentiating the moment of inertial J from (A.1.3) with respect to the time t,

$$\dot{J} = 2 \sum_{i=1}^{n} m_i \, \boldsymbol{r}_i \cdot \dot{\boldsymbol{r}}_i, \quad \ddot{J} = 2 \sum_{i=1}^{n} m_i \, \dot{\boldsymbol{r}}_i^2 + 2 \sum_{i=1}^{n} \boldsymbol{r}_i \cdot m_i \, \ddot{\boldsymbol{r}}_i,$$

where, replacing T from (A.1.3) and using (A.1.5), we obtain

$$\ddot{J} = 4\,T + 2 \sum_{i=1}^{n} \boldsymbol{r}_i \cdot \nabla_{\boldsymbol{r}_i} U.$$

Applying Proposition A.2 leads to

$$\ddot{J} = 4\,T + 2\,(\lambda + 1)\,U.$$

∎

Proposition A.4. *The moment of inertia J in terms of the mutual distances (Cid [7, Ch. III, p. III-7], Wintner [14, Ch. V, § 322bis, p. 243], Boccaletti and Pucacco [6, Ch. 3, § 3.3, Eq. (3.38), p. 193]) reads*

$$J = M^{-1} \sum_{1 \neq i < k \neq n} m_i \, m_k \, \|\boldsymbol{r}_{ik}\|^2, \quad \text{where} \quad M = \sum_{i=1}^{n} m_i. \tag{A.1.8}$$

Proposition A.5. *The center of mass of an n-body system of masses $m_1, \dots, m_n$ moves on a straight line with constant velocity.*

Proof. Denote $\boldsymbol{r}_{cm} = (\sum_i m_i \boldsymbol{r}_i) / (\sum_i m_i)$ the position vector of the *center of mass*. We want to prove that $\dot{\boldsymbol{r}}_{cm} = \boldsymbol{a}$, and $\boldsymbol{r}_{cm} = \boldsymbol{a}t + \boldsymbol{b}$, with $\boldsymbol{a}$, $\boldsymbol{b}$, constant vectors. Denoting $a_{ik} = m_i m_k \phi(|\boldsymbol{r}_{ik}|)$ in (A.1.1), adding up these equations over $i = 1, \dots, n$, and applying Proposition B.5 (Appendix B), we have

$$\sum_i m_i \ddot{\boldsymbol{r}}_i = \sum_i \sum_k m_i m_k \phi(\|\boldsymbol{r}_{ik}\|) \boldsymbol{r}_{ik} = \sum_i \sum_k a_{ik} \boldsymbol{r}_{ik} = \boldsymbol{0}.$$

Then, $\sum_i m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{0}$. Integrating we get $\sum_i m_i \dot{\boldsymbol{r}}_i = \boldsymbol{a}$, and $\sum_i m_i \boldsymbol{r}_i = \boldsymbol{a}t + \boldsymbol{b}$, with $\boldsymbol{a}$, $\boldsymbol{b}$ arbitrary constant vectors. In particular, in a barycentric reference system these constants are zero. ∎

Definition A.2. The *angular momentum* $\boldsymbol{G}$ of an n-body system is the vector

$$\boldsymbol{G} = \sum_{i=1}^{n} m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i. \tag{A.1.9}$$

Proposition A.6. *The angular momentum $\boldsymbol{G}$ of an n-body system is a constant vector.*

Proof. Forming the cross product of each equation from (A.1.1) with the respective vector $\boldsymbol{r}_i$, adding up over $i = 1, \dots, n$, and using Proposition B.5 (Appendix B),

$$\sum_i m_i \ddot{\boldsymbol{r}}_i \times \boldsymbol{r}_i = \sum_i \sum_k m_i m_k \phi(\|\boldsymbol{r}_{ik}\|) \boldsymbol{r}_{ik} \times \boldsymbol{r}_i = \sum_i \sum_k a_{ik} \boldsymbol{r}_{ik} \times \boldsymbol{r}_i = \sum_i \sum_k a_{ik} \boldsymbol{r}_k \times \boldsymbol{r}_i = \boldsymbol{0}.$$

Then, $\sum_i m_i \ddot{\boldsymbol{r}}_i \times \boldsymbol{r}_i = \boldsymbol{0}$, and, by virtue of the property $\mathrm{d}(\boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i)/\mathrm{d}t = \boldsymbol{r}_i \times \ddot{\boldsymbol{r}}_i$, we can integrate the last equality obtaining the result. ∎

Definition A.3. The *invariable plane* of the n-body problem is the plane passing through the center of mass of the system and perpendicular to its angular momentum vector. In other words, it is the set of points whose position vectors $\boldsymbol{r}$ satisfy the condition $\boldsymbol{G} \cdot \boldsymbol{r} = 0$.

Definition A.4. The *total energy* $\mathcal{E}$ of the system is the scalar function $\mathcal{E} = T - U$.

Proposition A.7. *The total energy of the n-body problem is constant.*

Proof. To prove it, differentiating U with respect to t and using (A.1.5), we obtain the following relation

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \nabla_{\boldsymbol{r}_i} U \cdot \dot{\boldsymbol{r}}_i = \sum_{i=1}^{n} m_i \ddot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i = \frac{\mathrm{d}T}{\mathrm{d}t} \implies \frac{\mathrm{d}(T-U)}{\mathrm{d}t} = 0 \implies T - U = \text{const.},$$

from which we conclude that T and U differ by an additive constant $\mathcal{E}$. ∎

To sum up, *the expressions from Propositions A.5, A.6 and A.7 provide us with the ten classical first integrals of the n-body problem, thanks to which the differential order of the system can be reduced from $6n$ to $6n - 10$.*

Appendix B

Auxiliary Results

B.1 Rotation matrices in $\mathbb{R}^3$

Definition B.1. A rotation matrix Q in $\mathbb{R}^3$ is a $3\,x\,3$ orthogonal matrix ($Q^{-1} = Q^T$) with $\det Q = 1$, i.e., $Q \in \mathrm{SO}(3)$.

Proposition B.1. *For any rotation matrix Q the following properties hold:*

$$\bullet\ (Q\,\boldsymbol{x}) \cdot (Q\,\boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{y}, \tag{B.1.1}$$

$$\bullet\ Q\,(\boldsymbol{x} \times \boldsymbol{y}) = (Q\,\boldsymbol{x}) \times (Q\,\boldsymbol{y}). \tag{B.1.2}$$

Proof.
• Using the general Proposition $\boldsymbol{x} \cdot M\boldsymbol{y} = M^T\boldsymbol{x} \cdot \boldsymbol{y}$, valid for any square matrix M, and considering that Q is orthogonal, $Q^{-1} = Q^T$, we have

$$(Q\,\boldsymbol{x}) \cdot (Q\,\boldsymbol{y}) = \left(Q^T Q\,\boldsymbol{x}\right) \cdot \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y}.$$

• Let $z \in \mathbb{R}^3$ be an arbitrary vector; then, there exists a vector $\tilde{z}$ such that $\tilde{z} = Q^{-1}\,z$, i.e., $z = Q\,\tilde{z}$. Then,

$$[(Q\,\boldsymbol{x}) \times (Q\,\boldsymbol{y})] \cdot z = [(Q\,\boldsymbol{x}) \times (Q\,\boldsymbol{y})] \cdot Q\,\tilde{z} = (\boldsymbol{x} \times \boldsymbol{y}) \cdot \tilde{z},$$

where we have applied the general property

$$[(M\,\boldsymbol{x}) \times (M\,\boldsymbol{y})] \cdot (M\,z) = (\det M)\ [(\boldsymbol{x} \times \boldsymbol{y}) \cdot z]\,,$$

valid for any $3x3$ matrix M.

In view of (B.1.1),

$$(\boldsymbol{x} \times \boldsymbol{y}) \cdot \tilde{z} = [Q\ (\boldsymbol{x} \times \boldsymbol{y})] \cdot Q\,\tilde{z} = [Q\ (\boldsymbol{x} \times \boldsymbol{y})] \cdot z,$$

and then,

$$[(Q\,\boldsymbol{x}) \times (Q\,\boldsymbol{y})] \cdot z = [Q\ (\boldsymbol{x} \times \boldsymbol{y})] \cdot z.$$

Taking into account that two vectors are equal if and only if their dot products with an arbitrary vector z are equal, the expression (B.1.2) is proved. ∎

Proposition B.2. *If Q is a rotation matrix of the class C^2 that depends on the time t, we can build an antisymmetric matrix W given by*

$$W = Q^{-1}\dot{Q} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \tag{B.1.3}$$

such that W^2 is a symmetric matrix and the following relation is satisfied:

$$Q^{-1}\ddot{Q} = \dot{W} + W^2. \tag{B.1.4}$$

Proof. Differentiating the equality $Q^{-1}Q = I_3$ and applying the orthogonality condition $Q^{-1} = Q^T$, we obtain

$$\frac{d}{dt}(Q^{-1}Q) = Q^{-1}\dot{Q} + \dot{Q}^{-1}Q = Q^{-1}\dot{Q} + \dot{Q}^T Q = 0_{3x3} \implies Q^{-1}\dot{Q} = -\dot{Q}^T Q. \tag{B.1.5}$$

Using (B.1.5), we prove that W is an antisymmetric matrix as follows:

$$W^T = (Q^{-1}\dot{Q})^T = -(\dot{Q}^T Q)^T = -Q^T\dot{Q} = -Q^{-1}\dot{Q} = -W.$$

It can be easily seen that W^2 is symmetric:

$$W^2 = W\,W = -\left(\dot{Q}^T\,Q\right)\left(Q^{-1}\dot{Q}\right) = -\dot{Q}^T\dot{Q} = -\left(\dot{Q}^T\dot{Q}\right)^T = (W^2)^T.$$

Finally, differentiating W in (B.1.3), Formula (B.1.4) is derived as follows:

$$\dot{W} = \frac{d}{dt}(Q^T\dot{Q}) = \dot{Q}^T\dot{Q} + Q^T\ddot{Q} = (Q\,W)^T\dot{Q} + Q^{-1}\ddot{Q} = -W^2 + Q^{-1}\ddot{Q} \implies Q^{-1}\ddot{Q} = \dot{W} + W^2.$$

■

Definition B.2. The *instantaneous angular velocity vector* corresponding to matrix W is the vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ that satisfies

$$W\boldsymbol{r} = \boldsymbol{\omega}\times\boldsymbol{r}, \qquad \boldsymbol{\omega} = \omega\,\boldsymbol{e}_\omega,$$

where $\boldsymbol{e}_\omega$ is a unit vector that represents the *direction of the instantaneous axis of rotation*, and $\omega = \|\boldsymbol{\omega}\|$ is the *norm of the angular velocity* with respect to the axis of rotation.

Proposition B.3. *The vector $\boldsymbol{e}_\omega$ fulfills the following properties:*

$$W\boldsymbol{e}_\omega = \boldsymbol{0}, \qquad W^2\,\boldsymbol{e}_\omega = \boldsymbol{0}. \tag{B.1.6}$$

Proof.

- $W\,\boldsymbol{e}_\omega = \boldsymbol{\omega}\times\boldsymbol{e}_\omega = (\omega\,\boldsymbol{e}_\omega)\times\boldsymbol{e}_\omega = \omega\,(\boldsymbol{e}_\omega\times\boldsymbol{e}_\omega) = \boldsymbol{0}.$ (B.1.7)
- $W^2\,\boldsymbol{e}_\omega = W\,(W\,\boldsymbol{e}_\omega) = W\,\boldsymbol{0} = \boldsymbol{0}.$ (B.1.8)

■

Proposition B.4. *The product of the matrices W and W^2 by any vector $\boldsymbol{x}\in\mathbb{R}^3$ is a vector orthogonal to the axis of rotation.*

Proof. Any vector $\boldsymbol{x}\in\mathbb{R}^3$ can be expressed as $\boldsymbol{x} = x_\omega\,\boldsymbol{e}_\omega + x_\pi\,\boldsymbol{e}_\pi$, where $\boldsymbol{e}_\pi$ represents a direction orthogonal to the angular velocity, i.e., it belongs to the instantaneous plane Π orthogonal to the axis of rotation given by the vector $\boldsymbol{e}_\omega$. Then,

$$W\,\boldsymbol{x} = x_\omega\,W\,\boldsymbol{e}_\omega + x_\pi\,W\,\boldsymbol{e}_\pi = \boldsymbol{0} + x_\pi\,(\boldsymbol{\omega}\times\boldsymbol{e}_\pi) = (x_\pi\,\omega)\,(\boldsymbol{e}_\omega\times\boldsymbol{e}_\pi) \implies W\boldsymbol{x}\perp\boldsymbol{e}_\omega.$$

Denoting $\boldsymbol{y} = W\,\boldsymbol{x}$ we have $W^2\,\boldsymbol{x} = W\,(W\,\boldsymbol{x}) = W\,\boldsymbol{y}\perp\boldsymbol{e}_\omega$. ■

B.2 Rotations and Kinematics

Let us suppose a vector $\boldsymbol{s} = \boldsymbol{s}(t)$ that describes the evolution of the position vector of a particle P given in an orthogonal reference frame $\mathcal{S}_1$. The product of a vector $\boldsymbol{s}$ by a scalar $\rho = \rho(t) > 0$ is a *dilatation* ($\rho > 1$) or a *contraction* ($\rho < 1$) of the norm of the position vector of P.

If the rotation matrix $Q = Q(t)$ formalizes the rotation that transforms the frame $\mathcal{S}_1$ into another orthogonal reference frame $\mathcal{S}_2$, then we can write

$$\boldsymbol{r} = \rho\,Q\,\boldsymbol{s}, \tag{B.2.1}$$

where $\boldsymbol{r} = \boldsymbol{r}(t)$ denotes the vector $\rho\, \boldsymbol{s}$ given in the reference frame $\mathcal{S}_2$.

The time derivatives of $\boldsymbol{r}$ expressed in the frame $\mathcal{S}_2$ are

$$\dot{\boldsymbol{r}} = \dot\rho\, Q\, \boldsymbol{s} + \rho\, \dot Q\, \boldsymbol{s} + \rho\, Q\, \dot{\boldsymbol{s}}, \qquad \ddot{\boldsymbol{r}} = \rho\, Q\, \ddot{\boldsymbol{s}} + \left(2\,\dot\rho\, Q + 2\,\rho\, \dot Q\right) \dot{\boldsymbol{s}} + \left(\ddot\rho\, Q + 2\,\dot\rho\, \dot Q + \rho \ddot Q\right) \boldsymbol{s}.$$

From (B.2.1), and applying the formulas (B.1.3) and (B.1.4), we obtain $\boldsymbol{r}$, $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ referred to $\mathcal{S}_1$:

$$Q^{-1}\boldsymbol{r} = \rho\, \boldsymbol{s}, \qquad Q^{-1}\dot{\boldsymbol{r}} = \mathcal{R}\, \boldsymbol{s} + \mathcal{I}\, \dot{\boldsymbol{s}}, \qquad Q^{-1}\ddot{\boldsymbol{r}} = \mathcal{K}\, \boldsymbol{s} + 2\,\mathcal{R}\, \dot{\boldsymbol{s}} + \mathcal{I}\, \ddot{\mathbf{s}}, \tag{B.2.2}$$

where

$$\begin{aligned}
\mathcal{I} &= \rho I_3 = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix}, \\
\mathcal{R} &= \dot\rho I_3 + \rho W = \begin{pmatrix} \dot\rho & -\rho\omega_3 & \rho\omega_2 \\ \rho\omega_3 & \dot\rho & -\rho\omega_1 \\ -\rho\omega_2 & \rho\omega_1 & \dot\rho \end{pmatrix}, \\
\mathcal{K} &= \ddot\rho I_3 + 2\dot\rho W + \rho \dot W + \rho W^2 \\
&= \begin{pmatrix} \ddot\rho - \rho(\omega_2^2 + \omega_3^2) & -2\dot\rho\omega_3 - \rho(\dot\omega_3 - \omega_1\omega_2) & 2\dot\rho\omega_2 + \rho(\dot\omega_2 + \omega_1\omega_2) \\ 2\dot\rho\omega_3 + \rho(\dot\omega_3 + \omega_1\omega_2) & \ddot\rho - \rho(\omega_1^2 + \omega_3^2) & -2\dot\rho\omega_1 - \rho(\dot\omega_1 - \omega_2\omega_3) \\ -2\dot\rho\omega_2 - \rho(\dot\omega_2 - \omega_1\omega_3) & 2\dot\rho\omega_1 + \rho(\dot\omega_1 + \omega_2\omega_3) & \ddot\rho - \rho(\omega_1^2 + \omega_2^2) \end{pmatrix}.
\end{aligned} \tag{B.2.3}$$

Let us consider now the special case in which the rotation from $\mathcal{S}_1$ to $\mathcal{S}_2$ is given by a rotation of angle $\theta = \theta(t)$ around the third axis. Then,

$$Q(t) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad W = \begin{pmatrix} 0 & -\dot\theta & 0 \\ \dot\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\omega} = (0, 0, \dot\theta).$$

$$\mathcal{K} = \begin{pmatrix} \ddot\rho - \rho\,\dot\theta^2 & -(\rho\,\ddot\theta + 2\,\dot\theta\,\dot\rho) & 0 \\ \rho\,\ddot\theta + 2\,\dot\theta\,\dot\rho & \ddot\rho - \rho\,\dot\theta^2 & 0 \\ 0 & 0 & \ddot\rho \end{pmatrix}. \tag{B.2.4}$$

B.3 Other results

Definition B.3. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of degree* $\lambda \in \mathbb{R}$ if

$$f(\alpha \boldsymbol{x}) = \alpha^\lambda f(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \mathbb{R}^n, \quad \forall \alpha \in \mathbb{R}\,. \tag{B.3.1}$$

Theorem B.1. Euler's Theorem for homogeneous functions. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous function of degree λ. Then,*

$$\boldsymbol{x} \cdot \nabla_{\boldsymbol{x}} f(\boldsymbol{x}) = \lambda\, f(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \mathbb{R}^n. \tag{B.3.2}$$

Proof. Let us take $\boldsymbol{x} \in \mathbb{R}^n$; define $\tilde{\boldsymbol{x}} = \alpha \boldsymbol{x}$ and the function $g(\alpha) = \alpha^\lambda f(\boldsymbol{x}) - f(\tilde{\boldsymbol{x}})$. Differentiating g with respect to α we obtain

$$g'(\alpha) = \lambda \alpha^{\lambda-1} f(\boldsymbol{x}) - \boldsymbol{x} \cdot \nabla_{\tilde{\boldsymbol{x}}} f(\tilde{\boldsymbol{x}}).$$

In particular for $\alpha = 1$ we have

$$g'(1) = \lambda\, f(\tilde{\boldsymbol{x}}) - \boldsymbol{x} \cdot \nabla_{\boldsymbol{x}} f(\boldsymbol{x}).$$

If f is a homogeneous functions of degree λ, then $g(\alpha) = 0$ and consequently $g'(\alpha) = g'(1) = 0$, which proves the theorem. ∎

Proposition B.5. *Given a finite set of scalars a_{ik} and vectors $\boldsymbol{x}_{ik}$, with $i, k = 1, 2, \ldots, n$, such that $a_{ik} = a_{ki}$ and $\boldsymbol{x}_{ik} = -\boldsymbol{x}_{ki}$, we have*

$$\sum_i \sum_k a_{ik} \boldsymbol{x}_{ik} = \mathbf{0}.$$

Proof.

$$\sum_i \sum_k a_{ik} \boldsymbol{x}_{ik} = \sum_{1 \le i < k \le n} a_{ik} \boldsymbol{x}_{ik} + \sum_{1 \le i < k \le n} a_{ik} \boldsymbol{x}_{ki} = \sum_{1 \le i < k \le n} a_{ik} (\boldsymbol{x}_{ik} - \boldsymbol{x}_{ik}) = \mathbf{0}.$$

∎